End-Term Study Guide for MH4714

N.B.
This list also applies to the Autumn I grade exam but
does not apply to the Autumn repeat examination.

*(You will be asked to answer three out of four questions.
Each question will be worth 20 marks.)*

1. Definition of $\lim_{n\to\infty} a_n = L$.
2. Prove that a convergent sequence is bounded.
3. Give an example of a bounded sequence which is not convergent.
4. Give an example,(and verify it,) of a strictly increasing or strictly decreasing sequence of rationals which converges to a rational.
   Give an example,(and verify it,) of a sequence of rationals which converges to an irrational. (See Questions 7 and 8 on Exercise Sheet IV, Question 1 on Exercise Sheet VI.)
5. State the *Completeness Axiom* for the real numbers.
6. Prove that there is a real number L such that $L^2 = 2$ (or $L^2 = 3$ or $L^3 = 5$ etc.)
7. Definition of $\lim_{x\to a} f(x) = L$.
8. When is a function said to be *continuous* at $a \in \mathbb{R}$?
9. Let $k \in \mathbb{R}$. Given that the functions $f(x) = x, g(x) = k$ are continuous over $\mathbb{R}$ and using the properties of limits, prove that every polynomial is also continuous over $\mathbb{R}$.
10. State the Intermediate Value Theorem.
11. Use the Intermediate Value Theorem to prove that a given equation has a real solution. (See Question 5 on Exercise Sheet VIII, Question 3 on Exercise Sheet X and Question 2 on Exercise Sheet XI.)
12. Give an example of a function which is bounded over an interval $[a, b]$ but which does not have a maximum value in $[a, b]$.
13. Give an example of a continuous function which is not bounded over an interval.
14. Give an example of a continuous function which is bounded over an interval but does not have a maximum value in the interval.

15. When is a function said to be *differentiable* at $a \in \mathbb{R}$?

16. Prove that, if f is differentiable at $a \in \mathbb{R}$, then f is continuous at a.

17. Give an example (and verify it) of a function which is continuous at a point $a \in \mathbb{R}$ but is not differentiable at a.

18. Give an example (and verify it) of a function which is differentiable once but not twice at a given point or give an example of a function which is differentiable twice but not three times at a point etc. (See Question 1 on Exercise Sheet X.)

19. Let f be continuous and differentiable over (a, b). Prove that, if f has a maximum or minimum value at $c \in (a, b)$, then $f'(c) = 0$.

20. Determine the maximum and minimum value of a function over an interval $[a, b]$ and justify your answer.(See Question 3 on Exercise Sheet VI, Question 5 on Exercise Sheet VIII, Question 1 on Exercise Sheet IX.)

21. State and prove Rolle's Theorem.

22. Give an example (and verify it) of a function $f(x)$ which is continuous over an interval $[a, b]$ with $f(a) = f(b)$ but where $f'(x)$ is not 0 anywhere in (a, b).

23. Use Rolle's theorem to prove the Mean Value Theorem.

24. Use the Intermediate Value Theorem and Rolle's theorem to prove that a given polynomial has exactly one real root. (See Question 1, Exercise Sheet XI.)

25. Use the Mean Value Theorem to prove that, if $f'(x) = 0$ for all x in an interval then $f(x)$ is a constant over the interval.

26. Show that the graph of f^{-1} is the reflection of the graph of f in the line $y = x$. (See Questions 2 and 3 on Exercise Sheet X.)

27. When is a function said to be integrable?

28. Give an example (and verify it) of a function which is not integrable.

29. Let $f(x) = x^2$ and let P_n be the partition of the interval $[a, b]$ into n equal subintervals.
    Verify that
    $$\lim_{n\to\infty} U(f, P_n) = \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3$$
    (See Question 3 on Sheet XI.)

30. Prove the Fundamental Theorem of Calculus, that is, if f be integrable over $[a, b]$ and $f = g'$ for some function g, then
    $$\int_a^b f(x)dx = g(b) - g(a).$$

31. Evaluate some area measures and definite integrals. (See Questions 2,4,5,6 on Exercise Sheet X and Question 2,4,5 on Exercise Sheet XI.)

32. Prove Taylor's Theorem.

33. Determine the Taylor expansion of a given function. (See Questions 6,7,8,9 on Exercise Sheet XI.)